

Dynamical degrees of Hurwitz correspondences

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ABSTRACT

Hurwitz correspondences are multivalued self-maps of the moduli space $\mathcal{M}_{0,N}$. They descend from single-valued holomorphic self-maps of Teichmüller space $\mathcal{T}(S^2, N)$, the universal cover of $\mathcal{M}_{0,N}$. We study the dynamics of Hurwitz correspondences via numerical invariants called *dynamical degrees*. For any Hurwitz correspondence \mathcal{H} , we show that $k \mapsto (k\text{th dynamical degree of } \mathcal{H})$ is a nonincreasing function of k . In particular, the largest dynamical degree of \mathcal{H} is a *Hurwitz number*.

1. Introduction

Dynamical degrees are numerical invariants associated to maps — regular or rational, single-valued or *multivalued* — from a smooth projective variety to itself. These invariants measure the dynamical complexity of iterating the given self-map. For g a regular map from X to itself, the k th dynamical degree of g is the spectral radius of g^* acting on the cohomology group $H^{k,k}(X)$. For g a rational map or rational multivalued map, we may not have $(g^n)^* = (g^*)^n$. The dynamical degrees of such g depend on the asymptotics of the pullback maps $(g^n)^*$.

To define dynamical degrees in this context, pick an ample divisor \mathfrak{h} on X . The k th dynamical degree of g is

$$\lim_{n \rightarrow \infty} \left(((g^n)^*(\mathfrak{h}^k)) \cdot (\mathfrak{h}^{\dim X - k}) \right)^{1/n}.$$

This limit exists and is independent of choice of ample divisor ([DS05, DS08, Tru15]). Dinh and Sibony ([DS05, DS08]) showed that the topological entropy of a rational map or rational multivalued map is bounded from above by the logarithm of the largest of its dynamical degrees.

We consider the dynamical degrees of a certain class of multivalued self-maps of the moduli spaces $\mathcal{M}_{0,\mathbf{P}}$. For \mathbf{P} a finite set, the variety $\mathcal{M}_{0,\mathbf{P}}$ parametrizes smooth genus zero curves with marked points indexed by \mathbf{P} . Let \mathcal{H} be a Hurwitz space parametrizing maps $f : C \rightarrow D$, where both C and D are smooth \mathbf{P} -marked genus zero curves, and f has specified branching behavior at and over the marked points on C and D respectively. \mathcal{H} has two maps to $\mathcal{M}_{0,\mathbf{P}}$: a “target curve” map π_1 and a “source curve” map π_2 . If \mathbf{P} contains all the branch values of f , then π_1 is a covering map, and $\pi_2 \circ \pi_1^{-1}$ defines a multivalued map — a **Hurwitz correspondence** — from $\mathcal{M}_{0,\mathbf{P}}$ to itself. We prove:

THEOREM 3.1. *Let \mathcal{H} be a Hurwitz correspondence on $\mathcal{M}_{0,\mathbf{P}}$. Denote by Θ_k its k th dynamical degree. Then $\Theta_0 \geq \dots \geq \Theta_{\dim \mathcal{M}_{0,\mathbf{P}}}$.*

COROLLARY 1.1. *The topological entropy of \mathcal{H} is less than or equal to Θ_0 , the topological degree of $\pi_1 : \mathcal{H} \rightarrow \mathcal{M}_{0,\mathbf{P}}$.*

It is known that the sequence of dynamical degrees of a rational map is log-concave. However, since the graphs of multivalued maps may be reducible, no pattern is expected for them in general (Section 2.2). Our proof of Theorem 3.1 relies heavily on the moduli space interpretations of the varieties in question. In [Gue05], Guedj studies the properties of rational maps whose topmost dynamical degree is their largest. Theorem 3.1 implies that this property is satisfied by the *inverse* of any Hurwitz correspondence. This inverse, if it exists, is generally multivalued as well, but under some conditions (Koch, [Koc13]), it is single-valued.

Note that $\mathcal{M}_{0,\mathbf{P}}$ is not compact. To make sense of the dynamical degrees of Hurwitz correspondences, we consider these correspondences as rational multivalued self-maps of projective compactifications of $\mathcal{M}_{0,\mathbf{P}}$. Since dynamical degrees are birational invariants (Dinh-Sibony [DS05, DS08] and Truong [Tru15]), the choice of compactification does not matter. To prove Theorem 3.1, we examine a given Hurwitz correspondence both on the stable curves compactification $\overline{\mathcal{M}}_{0,\mathbf{P}}$, and on $(|\mathbf{P}| - 3)$ -dimensional projective space.

Hurwitz correspondences were used by Koch ([Koc13]) to study the question of which branched covers $\phi : S^2 \rightarrow S^2$ are homotopic to rational functions $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Koch and Roeder ([KR15]) found that a certain class of Hurwitz correspondences, when considered on $\overline{\mathcal{M}}_{0,\mathbf{P}}$, have a property known as *algebraic stability*, and used this property to compute the dynamical degrees of correspondences in this class. Koch and Speyer, together with the author ([Ram15]), showed that *all* Hurwitz correspondences are algebraically stable on $\overline{\mathcal{M}}_{0,\mathbf{P}}$, which implies that their dynamical degrees can be computed as the largest eigenvalues of induced pushforward maps on the homology groups of $\overline{\mathcal{M}}_{0,\mathbf{P}}$. The author then showed ([Ram15]) that these pushforward maps can always be written as block-lower-triangular matrices, and that on the top-dimensional half of the homology groups, the largest eigenvalue is always in the topmost block.

This paper is intended to complement [Ram15] and to finally relate all the dynamical degrees of a given Hurwitz correspondence. Theorem 3.1 says in particular that the largest and therefore most important dynamical degree of \mathcal{H} is Θ_0 , the topological degree of the “target curve” map $\pi_1 : \mathcal{H} \rightarrow \mathcal{M}_{0,\mathbf{P}}$, hence an integer and a *Hurwitz number*. Hurwitz numbers count covers of \mathbb{P}^1 having specified branch locus on \mathbb{P}^1 and specified ramification profile. They also count the number of ways to factor the identity in the symmetric group S_d as a product of permutations $\sigma_1, \dots, \sigma_N$ with specified cycle types, that collectively generate a transitive subgroup. Thus the dynamically motivated quantity Θ_0 has a purely combinatorial interpretation.

The paper is organized as follows. Section 2 contains background on rational correspondences, the moduli space $\mathcal{M}_{0,\mathbf{P}}$ and its compactification $\overline{\mathcal{M}}_{0,\mathbf{P}}$, Hurwitz spaces, and Hurwitz correspondences. Section 3 contains the proof of Theorem 3.1. Section 4 is an Appendix containing a proof that dynamical degrees of rational correspondences are birational invariants.

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Conventions

All varieties and schemes are over \mathbb{C} . For X a variety, we denote by $\mathcal{Z}_k(X)$ the group of k -cycles on X ; that is, the free abelian group on the set of k -dimensional subvarieties of X . We denote by $A_k(X)$ the Chow group of k -cycles on X up to rational equivalence. For X a smooth variety, we denote by $A^k(X)$ the Chow group of codimension- k cycles on X .

2. Background

This section closely follows the background section of [Ram15].

2.1 Rational correspondences

Rational correspondences generalize the notion of a rational map. A rational correspondence from X to Y is a multivalued map to Y , defined on a dense open set of X .

DEFINITION 2.1 ([Ram15], Definition 4.1). Let X and Y be irreducible smooth projective varieties. A **rational correspondence** $(\Gamma, \pi_X, \pi_Y) : X \dashrightarrow Y$ is a diagram

$$\begin{array}{ccc} & \Gamma & \\ \pi_X \swarrow & & \searrow \pi_Y \\ X & & Y \end{array}$$

where Γ is a smooth quasiprojective variety, not necessarily irreducible, and the restriction of π_X to every irreducible component of Γ is dominant and generically finite.

Over a dense open set in X , π_X is a covering map, and $\pi_Y \circ \pi_X^{-1}$ defines a multivalued map to Y . However, considered as a multivalued map from X to Y , it is possible that $\pi_Y \circ \pi_X^{-1}$ has indeterminacy, since some fibers of π_X may be empty or positive-dimensional.

Like rational maps, rational correspondences induce pushforward and pullback maps of Chow groups, and can be composed with each other.

DEFINITION 2.2. Let $\bar{\Gamma}$ be a projective compactification of Γ such that Γ is dense in $\bar{\Gamma}$ and π_X and π_Y extend to maps $\bar{\pi}_X$ and $\bar{\pi}_Y$ defined on $\bar{\Gamma}$. The cycle $(\bar{\pi}_X \times \bar{\pi}_Y)_*[\bar{\Gamma}] \in \mathcal{Z}_{\dim X}(X \times Y)$ is independent of the choice of compactification $\bar{\Gamma}$, so we denote this cycle by $[\Gamma]$.

Remark 2.3. In [DS08], a rational correspondence from X to Y is defined as a cycle $\sum_i m_i [\Gamma_i]$ in $\mathcal{Z}_{\dim X}(X \times Y)$, such that each Γ_i maps surjectively onto X .

DEFINITION 2.4 ([Ram15], Section 4). Let $\bar{\Gamma}$ be a projective compactification of Γ as in Definition 2.2. Set

$$[\Gamma]_* := (\bar{\pi}_Y)_* \circ (\bar{\pi}_X)^* : A_k(X) \rightarrow A_k(Y)$$

and

$$[\Gamma]^* := (\bar{\pi}_X)_* \circ (\bar{\pi}_Y)^* : A^k(Y) \rightarrow A^k(X).$$

These pushforward and pullback maps are independent of the choice of compactification $\bar{\Gamma}$; they depend only on the cycle $[\Gamma]$ ([Ful98], Remark 6.2.2).

DEFINITION 2.5 ([Ram15], Section 4). Suppose $(\Gamma, \pi_X, \pi_Y) : X \dashrightarrow Y$ and $(\Gamma', \pi'_Y, \pi'_Z) : Y \dashrightarrow Z$ are rational correspondences such that the image under π_Y of every irreducible component of Γ intersects the domain of definition of the multivalued map $\pi'_Z \circ (\pi'_Y)^{-1}$. The **composite** $\Gamma' \circ \Gamma$ is a rational correspondence from X to Z defined as follows.

Pick dense open sets $U_X \subseteq X$ and $U_Y \subseteq Y$ such that $\pi_Y(\pi_X^{-1}(U_X)) \subseteq U_Y$, and $\pi_X|_{\pi_X^{-1}(U_X)}$ and $\pi'_Y|_{(\pi'_Y)^{-1}(U_Y)}$ are both covering maps. Set

$$\Gamma' \circ \Gamma := \pi_X^{-1}(U_X) \times_{\pi_Y \times \pi'_Y} (\pi'_Y)^{-1}(U_Y),$$

together with its given maps to X and Z .

This composite does depend on the choices of open sets U_X and U_Y , but the cycle $[\Gamma' \circ \Gamma]$ is well-defined. Note that $[\Gamma' \circ \Gamma]_*$ may not agree with $[\Gamma']_* \circ [\Gamma]_*$ and $[\Gamma' \circ \Gamma]^*$ may not agree with $[\Gamma]^* \circ [\Gamma']^*$.

2.2 Dynamical degrees

Dynamical degrees were first introduced as invariants of surjective holomorphic self-maps of a smooth projective variety. The k th dynamical degree of $g : X \rightarrow X$ is the spectral radius of $g^* : H^{k,k}(X) \rightarrow H^{k,k}(X)$. Gromov ([Gro03]) and Yomdin ([Yom87]) showed that the topological entropy of such a map is the logarithm of its largest dynamical degree. Dynamical degrees were later generalized to rational maps and rational correspondences. Dinh and Sibony ([DS05, DS08]) showed that the topological entropy of a rational map or correspondence is bounded from above by the logarithm of its largest dynamical degree.

DEFINITION 2.6 ([Ram15], Definition 4.3). Let $(\Gamma, \pi_1, \pi_2) : X \dashrightarrow X$ be a rational self-correspondence such that the restriction of π_2 to every irreducible component of Γ is dominant. In this case we say Γ is a **dominant** rational self-correspondence.

DEFINITION 2.7. Let Γ be as in Definition 2.6. Set $\Gamma^n := \Gamma \circ \cdots \circ \Gamma$ (n times), and pick \mathfrak{h} an ample divisor class on X . The k th **dynamical degree** Θ_k of Γ is defined to be

$$\lim_{n \rightarrow \infty} \left(([\Gamma^n]^*(\mathfrak{h}^k)) \cdot (\mathfrak{h}^{\dim X - k}) \right)^{1/n}.$$

This limit exists and is independent of choice of ample divisor ([DS05, DS08, Tru15]).

The dynamical degrees of Γ are determined by the cycle $[\Gamma]$.

THEOREM 2.8 (Birational invariance of dynamical degrees, [DS05, DS08, Tru15]). *Let $(\Gamma, \pi_1, \pi_2) : X \dashrightarrow X$ be a dominant rational self-correspondence, and let $\chi : X \dashrightarrow X'$ be a birational equivalence. We obtain a dominant rational self-correspondence on X' through conjugation by χ as follows. Let U be the domain of definition of χ , and set $\Gamma' = \pi_1^{-1}(U) \cap \pi_2^{-1}(U)$. We have a dominant rational self-correspondence*

$$(\Gamma', \chi \circ \pi_1, \chi \circ \pi_2) : X' \dashrightarrow X'.$$

Then the dynamical degrees of Γ and Γ' are equal.

REMARK 2.9. The references [DS05, DS08, Tru15] do not contain a proof of Theorem 2.8 as stated. Dinh and Sibony proved birational invariance of dynamical degrees of rational *maps* ([DS05]), and a different version of birational invariance for rational correspondences ([DS08]), both in the analytic setting. Truong ([Tru15]) reproved birational invariance for rational maps in the

algebraic setting. The proof in [Tru15] can be modified to obtain Theorem 2.8; see the Appendix for details.

The sequence of dynamical degrees of a rational map is log-concave. Let $g : X \dashrightarrow X$ be a dominant rational map, and let \mathfrak{h} be an ample divisor class on X . For $n > 0$ set $\text{Gr}(g^n)$ to be the graph of g^n in $X \times X$, with its two maps π_1^n and π_2^n to X . If Θ_k denotes the k th dynamical degree of g , we have

$$\begin{aligned} \Theta_k &= \lim_{n \rightarrow \infty} \left(((g^n)^*(\mathfrak{h}^k)) \cdot (\mathfrak{h}^{\dim X - k}) \right)^{1/n} \\ &= \lim_{n \rightarrow \infty} \left(((\pi_2^n)^*(\mathfrak{h}^k)) \cdot ((\pi_1^n)^*(\mathfrak{h}^{\dim X - k})) \right)^{1/n}, \end{aligned}$$

by the projection formula. Since $(\pi_2^n)^*(\mathfrak{h})$ and $(\pi_1^n)^*(\mathfrak{h})$ are nef on $\text{Gr}(g^n)$, and $\text{Gr}(g^n)$ is irreducible, the sequence of intersection numbers $\{((\pi_2^n)^*(\mathfrak{h}^k)) \cdot ((\pi_1^n)^*(\mathfrak{h}^{\dim X - k}))\}_k$ is log-concave ([Laz04], Example 1.6.4). Thus the sequence $\{\Theta_k\}_k$ is log-concave as well.

This argument breaks down for multivalued maps/rational correspondences since their graphs are not necessarily irreducible. Our proof of Theorem 3.1 deals separately with every irreducible component of every iterate of a given Hurwitz correspondence.

2.3 The moduli spaces $\mathcal{M}_{0,\mathbf{P}}$ and $\overline{\mathcal{M}}_{0,\mathbf{P}}$

The moduli space $\mathcal{M}_{0,\mathbf{P}}$ is a rational smooth quasiprojective variety parametrizing ways of marking \mathbb{P}^1 by elements of a finite set, up to projective change of coordinates.

DEFINITION 2.10. Let \mathbf{P} be a finite set. A **\mathbf{P} -marked smooth genus zero curve** is a curve C , isomorphic to \mathbb{P}^1 , together with an injective map $\iota : \mathbf{P} \hookrightarrow C$.

DEFINITION 2.11. Let $|\mathbf{P}| \geq 3$. There is a smooth quasiprojective variety $\mathcal{M}_{0,\mathbf{P}}$ of dimension $|\mathbf{P}| - 3$ parametrizing all \mathbf{P} -marked smooth genus zero curves up to isomorphism.

There are several compactifications of $\mathcal{M}_{0,\mathbf{P}}$ that extend its moduli space interpretation. The most widely-studied of these is the stable curves compactification $\overline{\mathcal{M}}_{0,\mathbf{P}}$. Another compactification is simply the projective space $\mathbb{P}^{|\mathbf{P}|-3}$.

DEFINITION 2.12. A **stable \mathbf{P} -marked genus zero curve** is a connected projective curve C of arithmetic genus zero whose only singularities are simple nodes, together with an injection $\iota : \mathbf{P} \rightarrow (\text{smooth locus of } C)$, such that the set of automorphisms $C \rightarrow C$ that commute with ι is finite.

THEOREM 2.13 (Deligne, Grothendieck, Knudsen, Mumford). *There is a smooth projective variety $\overline{\mathcal{M}}_{0,\mathbf{P}}$ of dimension $|\mathbf{P}| - 3$ that is a fine moduli space for stable \mathbf{P} -marked genus zero curves. It contains $\mathcal{M}_{0,\mathbf{P}}$ as a dense open subset.*

The complement $\overline{\mathcal{M}}_{0,\mathbf{P}} \setminus \mathcal{M}_{0,\mathbf{P}}$ is a simple normal crossings divisor, referred to as the **boundary** of $\overline{\mathcal{M}}_{0,\mathbf{P}}$. Given a subset $\mathbf{S} \subseteq \mathbf{P}$ such that $|\mathbf{S}|, |\mathbf{S}^C| \geq 2$, define a divisor $\delta_{\mathbf{S}} \subseteq \overline{\mathcal{M}}_{0,\mathbf{P}}$ as follows. Consider the locus of all $[C, \iota]$ in $\overline{\mathcal{M}}_{0,\mathbf{P}}$ such that C has two irreducible components joined at a node, the points $\iota(p)$ with $p \in \mathbf{S}$ are all on one component, and the points $\iota(p)$ with $p \in \mathbf{S}^C$ are all on the other component. Let $\delta_{\mathbf{S}}$ be the closure of this locus. Then $\delta_{\mathbf{S}}$ is an irreducible divisor contained in the boundary. All irreducible components of the boundary are obtained in this manner. Note that $\delta_{\mathbf{S}} = \delta_{\mathbf{S}^C}$.

DEFINITION 2.14. For an injection $j : \mathbf{P}' \hookrightarrow \mathbf{P}$ with $|\mathbf{P}'| \geq 3$, there is a **forgetful map** $\mu : \mathcal{M}_{0,\mathbf{P}} \rightarrow \mathcal{M}_{0,\mathbf{P}'}$ sending $[C, \iota]$ to $[C, \iota \circ j]$. This map extends to $\mu : \overline{\mathcal{M}}_{0,\mathbf{P}} \rightarrow \overline{\mathcal{M}}_{0,\mathbf{P}'}$.

The tautological ψ -classes. $\overline{\mathcal{M}}_{0,\mathbf{P}}$ has a tautological line bundle \mathcal{L}_p corresponding to each marked point $p \in \mathbf{P}$. This line bundle assigns to the point $[C, \iota]$ the 1-dimensional complex vector space $T_{\iota(p)}^\vee C$, namely, the cotangent line to the curve C at the marked point $\iota(p)$. The divisor class associated to \mathcal{L}_p is denoted ψ_p .

The space $H^0(\overline{\mathcal{M}}_{0,\mathbf{P}}, \mathcal{L}_p)$ is $(|\mathbf{P}| - 2)$ -dimensional and basepoint-free. The induced map $\rho : \overline{\mathcal{M}}_{0,\mathbf{P}} \rightarrow \mathbb{P}(H^0(\overline{\mathcal{M}}_{0,\mathbf{P}}, \mathcal{L}_p)^\vee) \cong \mathbb{P}^{|\mathbf{P}|-3}$ is a birational map onto $\mathbb{P}^{|\mathbf{P}|-3}$ ([Kap93]).

Consider a forgetful map $\mu : \overline{\mathcal{M}}_{0,\mathbf{P} \cup \{q\}} \rightarrow \overline{\mathcal{M}}_{0,\mathbf{P}}$. For $p \in \mathbf{P}$, we have ([AC98])

$$\mu^* \psi_p^{\overline{\mathcal{M}}_{0,\mathbf{P}}} = \psi_p^{\overline{\mathcal{M}}_{0,\mathbf{P} \cup \{q\}}} - \delta_{\{p,q\}}.$$

Using induction, we obtain:

LEMMA 2.15. For a forgetful map $\mu : \overline{\mathcal{M}}_{0,\mathbf{P} \cup \mathbf{Q}} \rightarrow \overline{\mathcal{M}}_{0,\mathbf{P}}$, we have

$$\mu^* \psi_p^{\overline{\mathcal{M}}_{0,\mathbf{P}}} = \psi_p^{\overline{\mathcal{M}}_{0,\mathbf{P} \cup \mathbf{Q}}} - \sum_{\substack{\mathbf{S} \subseteq \mathbf{Q} \\ \mathbf{S} \text{ nonempty}}} \delta_{\{p\} \sqcup \mathbf{S}}.$$

2.4 Hurwitz spaces and Hurwitz correspondences

Hurwitz spaces are moduli spaces parametrizing finite maps with prescribed ramification between smooth curves. See [RW06] for a summary.

DEFINITION 2.16. A **partition** λ of a positive integer k is a multiset of positive integers whose sum with multiplicity is k .

DEFINITION 2.17. A multiset λ_1 is a **submultiset** of λ_2 if for all $r \in \lambda_1$, the multiplicity of occurrence of r in λ_1 is less than or equal to the multiplicity of occurrence of r in λ_2 .

DEFINITION 2.18 (*Hurwitz space*, [Ram15], Definition 5.4). Fix discrete data:

- \mathbf{A} and \mathbf{B} finite sets with cardinality at least 3 (marked points on source and target curves, respectively),
- d a positive integer (degree),
- $F : \mathbf{A} \rightarrow \mathbf{B}$ a map,
- $\text{br} : \mathbf{B} \rightarrow \{\text{partitions of } d\}$ (branching), and
- $\text{rm} : \mathbf{A} \rightarrow \mathbb{Z}^{>0}$ (ramification),

such that

- (Condition 1, Riemann-Hurwitz constraint) $\sum_{b \in \mathbf{B}} (d - \text{length of } \text{br}(b)) = 2d - 2$, and
- (Condition 2) for all $b \in \mathbf{B}$, the multiset $(\text{rm}(a))_{a \in F^{-1}(b)}$ is a submultiset of $\text{br}(b)$.

There exists a smooth quasiprojective variety $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$, a **Hurwitz space**, parametrizing morphisms $f : C \rightarrow D$ up to isomorphism, where

- C and D are \mathbf{A} -marked and \mathbf{B} -marked smooth connected genus zero curves, respectively,
- f is degree d ,
- for all $a \in \mathbf{A}$, $f(a) = F(a)$ (using the injections $\mathbf{A} \hookrightarrow C$ and $\mathbf{B} \hookrightarrow D$),

- for all $b \in \mathbf{B}$, the branching of f over b is given by the partition $\text{br}(b)$, and
- for all $a \in \mathbf{A}$, the local degree of f at a is equal to $\text{rm}(a)$.

The Hurwitz space \mathcal{H} has a “source curve” map $\pi_{\mathbf{A}}$ to $\mathcal{M}_{0,\mathbf{A}}$ sending $[f : C \rightarrow D]$ to the marked curve $[C]$. There is similarly a “target curve” map $\pi_{\mathbf{B}}$ from \mathcal{H} to $\mathcal{M}_{0,\mathbf{B}}$. Unless \mathcal{H} is empty, $\pi_{\mathbf{B}}$ is a finite covering map. Thus for smooth compactifications $X_{\mathbf{A}}$ of $\mathcal{M}_{0,\mathbf{A}}$ and $X_{\mathbf{B}}$ of $\mathcal{M}_{0,\mathbf{B}}$, $(\mathcal{H}, \pi_{\mathbf{B}}, \pi_{\mathbf{A}}) : X_{\mathbf{B}} \rightrightarrows X_{\mathbf{A}}$ is a rational correspondence. We generalize this notion.

DEFINITION 2.19 (*Hurwitz correspondence*, [Ram15], Definition 5.5). Let \mathbf{A}' be any subset of \mathbf{A} with cardinality at least 3. There is a forgetful morphism $\mu : \mathcal{M}_{0,\mathbf{A}} \rightarrow \mathcal{M}_{0,\mathbf{A}'}$. Let Γ be a union of connected components of \mathcal{H} . If $X_{\mathbf{A}'}$ and $X_{\mathbf{B}}$ are smooth projective compactifications of $\mathcal{M}_{0,\mathbf{A}'}$ and $\mathcal{M}_{0,\mathbf{B}}$ respectively, then

$$(\Gamma, \pi_{\mathbf{B}}, \mu \circ \pi_{\mathbf{A}}) : X_{\mathbf{B}} \rightrightarrows X_{\mathbf{A}'}$$

is a rational correspondence. We call such a rational correspondence a ***Hurwitz correspondence***.

Consider a Hurwitz space $\mathcal{H} = \mathcal{H}(\mathbf{P}', \mathbf{P}, d, F, \text{br}, \text{rm})$ together with an injection $\mathbf{P} \hookrightarrow \mathbf{P}'$. This induces a forgetful map $\mu : \mathcal{M}_{0,\mathbf{P}'} \rightarrow \mathcal{M}_{0,\mathbf{P}}$. For Γ any union of connected components of \mathcal{H} , we have the “target curve” map $\pi_1 = \pi_{\mathbf{P}}|_{\Gamma} : \Gamma \rightarrow \mathcal{M}_{0,\mathbf{P}}$ and the “source curve” map $\pi_2 = \mu \circ \pi_{\mathbf{P}'}|_{\Gamma} : \Gamma \rightarrow \mathcal{M}_{0,\mathbf{P}}$. This defines a multivalued map from $\mathcal{M}_{0,\mathbf{P}}$ to itself and a rational self-correspondence $(\Gamma, \pi_1, \pi_2) : X_{\mathbf{P}} \rightrightarrows X_{\mathbf{P}}$ on any compactification $X_{\mathbf{P}}$ of $\mathcal{M}_{0,\mathbf{P}}$. These Hurwitz self-correspondences arise in the context of complex dynamics on \mathbb{P}^1 . (Koch, see [Koc13] or [Ram15] for a summary.) Note that by Theorem 2.8, the dynamical degrees of the Hurwitz self-correspondence Γ do not depend on the choice of compactification $X_{\mathbf{P}}$.

2.5 Fully marked Hurwitz spaces and admissible covers

Harris and Mumford ([HM82]) constructed compactifications of Hurwitz spaces. These compactifications are called moduli spaces of ***admissible covers***. They are projective varieties that parametrize certain ramified maps between nodal curves. They extend the “target curve” and “source curve” maps to the stable curves compactifications of the moduli spaces of target and source curves, respectively.

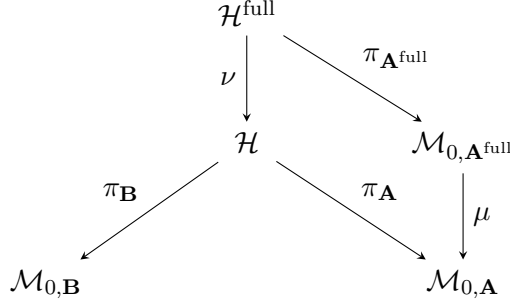
In general, the admissible covers compactifications are only coarse moduli spaces with *orbifold singularities*. For technical ease, we introduce a class of Hurwitz spaces whose admissible covers compactifications are fine moduli spaces. We call these Hurwitz spaces ***fully marked***.

DEFINITION 2.20 ([Ram15], Definition 5.6). Given $(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ as in Definition 2.18 with Condition 2 strengthened to:

- (Condition 2') For all $b \in \mathbf{B}$, the multiset $(\text{rm}(a))_{a \in F^{-1}(b)}$ is **equal to** $\text{br}(b)$,

we refer to the corresponding Hurwitz space $\mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ as a ***fully marked Hurwitz space***.

Given any Hurwitz space $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$, there exists a fully marked Hurwitz space $\mathcal{H}^{\text{full}} = \mathcal{H}(\mathbf{A}^{\text{full}}, \mathbf{B}, d, F, \text{br}, \text{rm})$, where \mathbf{A}^{full} is a superset of \mathbf{A} extending the functions F and rm . There is a finite covering map $\nu : \mathcal{H}^{\text{full}} \rightarrow \mathcal{H}$, and we have the following commutative diagram (see [Ram15] for details):



For Γ a union of connected components of \mathcal{H} , and for $X_{\mathbf{B}}$ and $X_{\mathbf{A}}$ smooth projective compactifications of $\mathcal{M}_{0,\mathbf{B}}$ and $\mathcal{M}_{0,\mathbf{A}}$, respectively, $(\Gamma, \pi_{\mathbf{B}}, \pi_{\mathbf{A}}) : X_{\mathbf{B}} \rightrightarrows X_{\mathbf{A}}$ is a Hurwitz correspondence. Set $\Gamma^{\text{full}} = \nu^{-1}(\Gamma)$. Then Γ^{full} is a union of connected components of $\mathcal{H}^{\text{full}}$, and in $\mathcal{Z}_{\dim X_{\mathbf{B}}}(X_{\mathbf{B}} \times X_{\mathbf{A}})$,

$$[\Gamma] = \frac{1}{\deg \nu} [\Gamma^{\text{full}}].$$

LEMMA 2.21. *Let $(\Gamma, \pi_1, \pi_2) : X_{\mathbf{P}} \rightrightarrows X_{\mathbf{P}}$ be a dominant Hurwitz self-correspondence. Then*

$$(k\text{th dynamical degree of } \Gamma) = \frac{1}{\deg \nu} (k\text{th dynamical degree of } \Gamma^{\text{full}}),$$

where Γ^{full} is a union of connected components of a fully marked Hurwitz space $\mathcal{H}^{\text{full}}$ corresponding to a superset \mathbf{P}^{full} of \mathbf{P} , and $\nu : \Gamma^{\text{full}} \rightarrow \Gamma$ is a finite covering map.

Proof. For Γ^{full} as above, we have that for every iterate Γ^n ,

$$[\Gamma^n] = \left(\frac{1}{\deg \nu} \right)^n [(\Gamma^{\text{full}})^n]. \quad \square$$

This means that arbitrary Hurwitz correspondences may be studied via fully marked Hurwitz spaces. These in turn have convenient compactifications by spaces of admissible covers.

THEOREM 2.22 (Harris and Mumford, [HM82]). *Given $(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ satisfying Conditions 1 and 2' as in Definition 2.20, there is a projective variety $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ containing $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ as a dense open subset. This admissible covers compactification $\overline{\mathcal{H}}$ extends the maps $\pi_{\mathbf{B}}$ and $\pi_{\mathbf{A}}$ to maps $\overline{\pi}_{\mathbf{B}}$ and $\overline{\pi}_{\mathbf{A}}$ to $\overline{\mathcal{M}}_{0,\mathbf{B}}$ and $\overline{\mathcal{M}}_{0,\mathbf{A}}$, respectively, with $\overline{\pi}_{\mathbf{B}} : \overline{\mathcal{H}} \rightarrow \overline{\mathcal{M}}_{0,\mathbf{B}}$ a finite flat map. $\overline{\mathcal{H}}$ may not be normal, but its normalization is smooth.*

PROPOSITION 2.23 (Ionel, [Ion01]). *Let $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ be a fully marked space of admissible covers with maps $\overline{\pi}_{\mathbf{B}}$ and $\overline{\pi}_{\mathbf{A}}$ to $\overline{\mathcal{M}}_{0,\mathbf{B}}$ and $\overline{\mathcal{M}}_{0,\mathbf{A}}$ respectively. Suppose we have $a \in \mathbf{A}$ and $b \in \mathbf{B}$ with $F(a) = b$. Then $(\overline{\pi}_{\mathbf{B}})^*(\mathcal{L}_b) = (\overline{\pi}_{\mathbf{A}})^*(\mathcal{L}_a)^{\otimes \text{rm}(a)}$ as line bundles on $\overline{\mathcal{H}}$.*

3. Main Theorem

THEOREM 3.1. *Let $(\Gamma, \pi_1, \pi_2) : \overline{\mathcal{M}}_{0,\mathbf{P}} \rightrightarrows \overline{\mathcal{M}}_{0,\mathbf{P}}$ be a dominant Hurwitz self-correspondence, and let Θ_k be the k th dynamical degree of Γ . Then*

$$\Theta_0 \geq \Theta_1 \geq \cdots \geq \Theta_{|\mathbf{P}|-3}.$$

Proof. By Lemma 2.21, we may assume Γ is a union of connected components of a fully marked Hurwitz space $\mathcal{H} = \mathcal{H}(\mathbf{P}^{\text{full}}, \mathbf{P}, d, F, \text{br}, \text{rm})$ corresponding to a superset \mathbf{P}^{full} of \mathbf{P} . Let $\overline{\mathcal{H}}$ denote

the admissible covers compactification of \mathcal{H} , and let $\overline{\Gamma}$ be the closure of Γ in $\overline{\mathcal{H}}$. For $\ell > 0$ set Γ^ℓ to be the ℓ th iterate of Γ , that is

$$\Gamma \xrightarrow{\pi_2 \times \pi_1} \cdots \xrightarrow{\pi_2 \times \pi_1} \Gamma \quad (\ell \text{ times}),$$

Set $\overline{\Gamma}^\ell$ to be its compactification

$$\overline{\Gamma} \xrightarrow{\pi_2 \times \pi_1} \cdots \xrightarrow{\pi_2 \times \pi_1} \overline{\Gamma} \quad (\ell \text{ times}),$$

with $\overline{\pi}_1^\ell$ and $\overline{\pi}_2^\ell$ its two maps to $\overline{\mathcal{M}}_{0,\mathbf{P}}$.

Since $\overline{\pi}_1^\ell$ is a flat map, no irreducible component of $\overline{\Gamma}^\ell$ is supported over the boundary of $\overline{\mathcal{M}}_{0,\mathbf{P}}$. This means that Γ^ℓ is a dense open subset of $\overline{\Gamma}^\ell$. We refer to the complement $\overline{\Gamma}^\ell \setminus \Gamma^\ell$ as the boundary of $\overline{\Gamma}^\ell$. The inverse image under $\overline{\pi}_1^\ell$ of the boundary of $\overline{\mathcal{M}}_{0,\mathbf{P}}$ is exactly the boundary of $\overline{\Gamma}^\ell$. The inverse image under $\overline{\pi}_2^\ell$ of the boundary of $\overline{\mathcal{M}}_{0,\mathbf{P}}$ is contained in the boundary of $\overline{\Gamma}^\ell$.

The compactification $\overline{\Gamma}^\ell$ is singular. However, for Cartier divisors $D_1, \dots, D_{\dim \overline{\Gamma}^\ell}$, the intersection product $D_1 \cdots D_{\dim \overline{\Gamma}^\ell}$ is a well-defined integer as in Section 1.1.C of [Laz04]. For any subscheme Y of dimension k , and Cartier divisors D_1, \dots, D_k , we similarly have the intersection number $D_1 \cdots D_k \cdot Y \in \mathbb{Z}$.

LEMMA 3.2. *For all $p \in \mathbf{P}$ and for all $\ell \geq 0$, there is an equality of Cartier divisors on $\overline{\Gamma}^\ell$ of the form*

$$(\overline{\pi}_1^\ell)^*(\psi_{F^\ell(p)}) = r \cdot (\overline{\pi}_2^\ell)^*(\psi_p) + E,$$

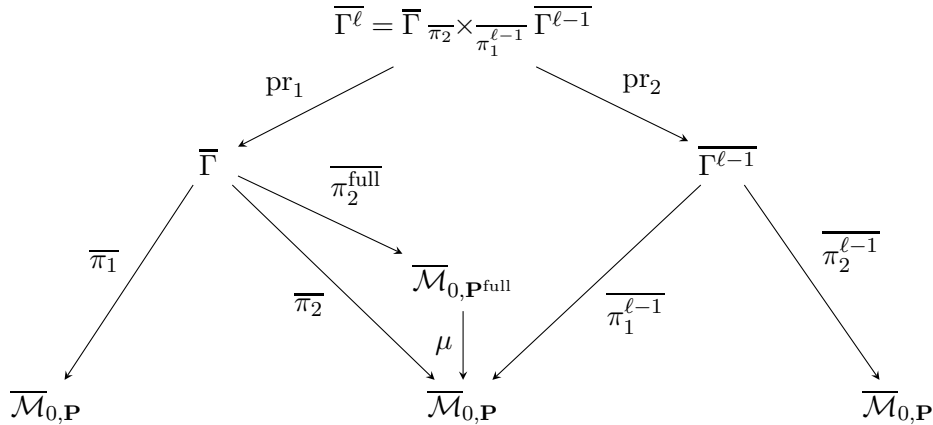
where r is a positive integer and E is an effective Cartier divisor supported on the boundary of $\overline{\Gamma}^\ell$.

Proof. We induct on ℓ . By convention, $\overline{\Gamma}^0$ is the identity rational correspondence

$$(\overline{\mathcal{M}}_{0,\mathbf{P}}, \overline{\pi}_1^0 = \text{Id}, \overline{\pi}_2^0 = \text{Id}) : \overline{\mathcal{M}}_{0,\mathbf{P}} \dashrightarrow \overline{\mathcal{M}}_{0,\mathbf{P}}.$$

For all $p \in \mathbf{P}$, $F^0(p) = p$, so $(\overline{\pi}_1^0)^*(\psi_{F^0(p)}) = (\overline{\pi}_2^0)^*(\psi_p)$. This gives us the base case $\ell = 0$.

Suppose the Lemma holds for $\ell - 1$. We have



For all $p \in \mathbf{P}$, we have

$$\begin{aligned} (\overline{\pi_1^\ell})^*(\psi_{F^\ell(p)}) &= \text{pr}_1^*(\overline{\pi_1})^*(\psi_{F^\ell(p)}) \\ &= \text{pr}_1^*\left(\text{rm}(F^{\ell-1}(p)) \cdot (\overline{\pi_2^{\text{full}}})^*(\psi_{F^{\ell-1}(p)}^{\mathbf{P}^{\text{full}}})\right) \quad (\text{by Proposition 2.23}). \end{aligned}$$

By Lemma 2.15,

$$\psi_{F^{\ell-1}(p)}^{\mathbf{P}^{\text{full}}} = \mu^*(\psi_{F^{\ell-1}(p)}) + \sum_{\mathbf{S} \subseteq \mathbf{P}^{\text{full}} \setminus \mathbf{P}} \delta_{\{F^{\ell-1}(p)\} \cup \mathbf{S}}.$$

The inverse image under $\overline{\pi_2^{\text{full}}}$ of the boundary in $\overline{\mathcal{M}}_{0,\mathbf{P}^{\text{full}}}$ is contained in the boundary of $\overline{\Gamma}$ (in fact it is the entire boundary), and the inverse image under pr_1 of the boundary of $\overline{\Gamma}$ is the boundary of $\overline{\Gamma}^\ell$. Thus, the Cartier divisor

$$E_1 := \text{pr}_1^* \left((\overline{\pi_2^{\text{full}}})^* \left(\sum_{\mathbf{S} \subseteq \mathbf{P}^{\text{full}} \setminus \mathbf{P}} \delta_{\{F^{\ell-1}(p)\} \cup \mathbf{S}} \right) \right)$$

is effective and supported on the boundary of $\overline{\Gamma}^\ell$. Set $r_1 = \text{rm}(F^{\ell-1}(p))$. We continue:

$$\begin{aligned} (\overline{\pi_1^\ell})^*(\psi_{F^\ell(p)}) &= r_1 \text{pr}_1^*(\overline{\pi_2^{\text{full}}})^* \mu^*(\psi_{F^{\ell-1}(p)}) + r_1 E_1 \\ &= r_1 \text{pr}_1^*(\overline{\pi_2})^*(\psi_{F^{\ell-1}(p)}) + r_1 E_1 \\ &= r_1 \text{pr}_2^*(\overline{\pi_1^{\ell-1}})^*(\psi_{F^{\ell-1}(p)}) + r_1 E_1. \end{aligned}$$

By the inductive hypothesis, we can rewrite this as

$$r_1 \text{pr}_2^*(r_2 (\overline{\pi_2^{\ell-1}})^*(\psi_p) + E_2) + r_1 E_1,$$

where r_2 is a positive integer and E_2 is an effective Cartier divisor supported on the boundary of $\overline{\Gamma}^{\ell-1}$. Since the inverse image under pr_2 of the boundary of $\overline{\Gamma}^{\ell-1}$ is contained in the boundary of $\overline{\Gamma}^\ell$, $\text{pr}_2^*(E_2)$ is an effective Cartier divisor supported on the boundary of $\overline{\Gamma}^\ell$. Thus we can finally write

$$\begin{aligned} (\overline{\pi_1^\ell})^*(\psi_{F^\ell(p)}) &= r_1 r_2 \text{pr}_2^*(\overline{\pi_2^{\ell-1}})^*(\psi_p) + r_1 \text{pr}_2^*(E_2) + r_1 E_1 \\ &= (r_1 r_2) (\overline{\pi_2^\ell})^*(\psi_p) + (r_1 \text{pr}_2^*(E_2) + r_1 E_1), \end{aligned}$$

which is as desired. This proves Lemma 3.2. \square

Now, since $F : \mathbf{P} \rightarrow \mathbf{P}$ is a map of finite sets, every point is eventually periodic. Pick $p \in \mathbf{P}$ with $F^\ell(p) = p$ for some $\ell > 0$. Then for every multiple $m\ell$, we have on $\overline{\Gamma}^{m\ell}$:

$$(\overline{\pi_1^{m\ell}})^*(\psi_p) = r_m (\overline{\pi_2^{m\ell}})^*(\psi_p) + E_m,$$

where r_m is a positive integer and E_m is an effective Cartier divisor supported on the boundary of $\overline{\Gamma}^{m\ell}$. Thus, if Y is a curve on $\overline{\Gamma}^{m\ell}$ with no irreducible component contained in the boundary, we have

$$(\overline{\pi_1^{m\ell}})^*(\psi_p) \cdot Y = r_m (\overline{\pi_2^{m\ell}})^*(\psi_p) \cdot Y + E_m \cdot Y$$

Since $(\overline{\pi_1^{m\ell}})^*(\psi_p)$ and $(\overline{\pi_2^{m\ell}})^*(\psi_p)$ are nef on $\overline{\Gamma}^{m\ell}$, these intersection numbers are nonnegative and we obtain:

$$(\overline{\pi_1^{m\ell}})^*(\psi_p) \cdot Y \geq (\overline{\pi_2^{m\ell}})^*(\psi_p) \cdot Y. \quad (1)$$

Let $N = |\mathbf{P}|$. Let $\rho : \overline{\mathcal{M}}_{0,\mathbf{P}} \rightarrow \mathbb{P}^{N-3}$ be the birational morphism to projective space given by the line bundle \mathcal{L}_p . Let \mathfrak{h} be the Cartier divisor class of a hyperplane in \mathbb{P}^{N-3} . Then $\rho^*(\mathfrak{h}) = \psi_p$.

The pullback $[\Gamma^n]^*(\mathfrak{h}^k)$ is by definition

$$(\rho \circ \overline{\pi_1^n})_* \circ (\rho \circ \overline{\pi_2^n})^*(\mathfrak{h}^k).$$

So, by the projection formula,

$$([\Gamma^n]^*(\mathfrak{h}^k)) \cdot (\mathfrak{h}^{N-3-k}) = ((\rho \circ \overline{\pi_2^n})^*(\mathfrak{h}^k)) \cdot ((\rho \circ \overline{\pi_1^n})^*(\mathfrak{h}^{N-3-k})).$$

Since dynamical degrees are birational invariants, Θ_k is also the k th dynamical degree of the induced rational correspondence $(\Gamma, \rho \circ \overline{\pi_1}, \rho \circ \overline{\pi_2}) : \mathbb{P}^{N-3} \dashrightarrow \mathbb{P}^{N-3}$. We have

$$\begin{aligned} \Theta_k &= \lim_{n \rightarrow \infty} (([\Gamma^n]^*(\mathfrak{h}^k)) \cdot (\mathfrak{h}^{N-3-k}))^{1/n} \\ &= \lim_{n \rightarrow \infty} (((\rho \circ \overline{\pi_2^n})^*(\mathfrak{h}^k)) \cdot ((\rho \circ \overline{\pi_1^n})^*(\mathfrak{h}^{N-3-k})))^{1/n} \\ &= \lim_{n \rightarrow \infty} (((\overline{\pi_2^n})^*(\psi_p^k)) \cdot ((\overline{\pi_1^n})^*(\psi_p^{N-3-k})))^{1/n}. \end{aligned}$$

Since this sequence converges, we can find its limit using any subsequence, and

$$\Theta_k = \lim_{m \rightarrow \infty} (((\overline{\pi_2^{m\ell}})^*(\psi_p^k)) \cdot ((\overline{\pi_1^{m\ell}})^*(\psi_p^{N-3-k})))^{1/m\ell} = \lim_{m \rightarrow \infty} (((\overline{\pi_2^{m\ell}})^*(\psi_p))^k \cdot ((\overline{\pi_1^{m\ell}})^*(\psi_p))^{N-3-k})^{1/m\ell}.$$

LEMMA 3.3. *Fix $m > 0$. The intersection numbers*

$$\alpha_k := ((\overline{\pi_2^{m\ell}})^*(\psi_p))^k \cdot ((\overline{\pi_1^{m\ell}})^*(\psi_p))^{N-3-k}$$

on $\overline{\Gamma^{m\ell}}$ are a nonincreasing function of k .

Proof of Lemma 3.3. We first show that for every irreducible component $\overline{\mathcal{J}}$ of $\overline{\Gamma^{m\ell}}$, the intersection numbers

$$\alpha_{\overline{\mathcal{J}},k} := ((\overline{\pi_2^{m\ell}})^*(\psi_p))|_{\overline{\mathcal{J}}}^k \cdot ((\overline{\pi_1^{m\ell}})^*(\psi_p))|_{\overline{\mathcal{J}}}^{N-3-k}$$

are a nonincreasing function of k .

Fix $\overline{\mathcal{J}}$ such an irreducible component. Since $(\overline{\pi_1^{m\ell}})^*(\psi_p)$ and $(\overline{\pi_2^{m\ell}})^*(\psi_p)$ are pullbacks of the ample hyperplane class \mathfrak{h} , they are nef on $\overline{\Gamma^{m\ell}}$ and $\overline{\mathcal{J}}$. So, $\alpha_{\overline{\mathcal{J}},k}$ is a log-concave function of k ([Laz04], Example 1.6.4). It therefore suffices to show that

$$\alpha_{\overline{\mathcal{J}},0} = ((\overline{\pi_1^{m\ell}})^*(\psi_p))|_{\overline{\mathcal{J}}}^{N-3} \geq ((\overline{\pi_2^{m\ell}})^*(\psi_p))|_{\overline{\mathcal{J}}}^1 \cdot ((\overline{\pi_1^{m\ell}})^*(\psi_p))|_{\overline{\mathcal{J}}}^{N-4} = \alpha_{\overline{\mathcal{J}},1}.$$

Note that $\psi_p^{N-4} = \rho^*(\mathfrak{h}^{N-4})$. The class \mathfrak{h}^{N-4} on \mathbb{P}^{N-3} may be represented by a line L that does not intersect the codimension-two exceptional locus of ρ . Then $\rho^{-1}(L)$ is an irreducible curve in $\overline{\mathcal{M}}_{0,\mathbf{P}}$ not contained in the boundary and $(\overline{\pi_1^{m\ell}})^{-1}(\rho^{-1}(L))|_{\overline{\mathcal{J}}}$ is a curve Y none of whose irreducible components lies in the boundary of $\overline{\mathcal{J}}$. Since $\overline{\pi_1^{m\ell}}$ is a flat map, and a covering map away from the boundary,

$$((\overline{\pi_1^{m\ell}})^*(\psi_p^{N-4}))|_{\overline{\mathcal{J}}} = [Y].$$

Thus,

$$\begin{aligned}
 \alpha_{\overline{\mathcal{J}},0} &= ((\overline{\pi_1^{m\ell}})^*(\psi_p))|_{\overline{\mathcal{J}}}^{N-3} = ((\overline{\pi_1^{m\ell}})^*(\psi_p))|_{\overline{\mathcal{J}}} \cdot ((\overline{\pi_1^{m\ell}})^*(\psi_p^{N-4}))|_{\overline{\mathcal{J}}} \\
 &= ((\overline{\pi_1^{m\ell}})^*(\psi_p))|_{\overline{\mathcal{J}}} \cdot Y \\
 &\geq ((\overline{\pi_2^{m\ell}})^*(\psi_p))|_{\overline{\mathcal{J}}} \cdot Y \quad (\text{by (1)}) \\
 &= ((\overline{\pi_2^{m\ell}})^*(\psi_p))|_{\overline{\mathcal{J}}} \cdot ((\overline{\pi_1^{m\ell}})^*(\psi_p^{N-4}))|_{\overline{\mathcal{J}}} \\
 &= ((\overline{\pi_2^{m\ell}})^*(\psi_p))|_{\overline{\mathcal{J}}} \cdot ((\overline{\pi_1^{m\ell}})^*(\psi_p))|_{\overline{\mathcal{J}}}^{N-4} = \alpha_{\overline{\mathcal{J}},1}.
 \end{aligned}$$

We conclude that for fixed $\overline{\mathcal{J}}$, $\alpha_{\overline{\mathcal{J}},k}$ is a nonincreasing function of k .

Thus $\alpha_k = \sum_{\overline{\mathcal{J}}} \alpha_{\overline{\mathcal{J}},k}$ is a nonincreasing function of k . \square

We now complete the proof of Theorem 3.1. For all m ,

$$((\overline{\pi_2^{m\ell}})^*(\psi_p))^k \cdot ((\overline{\pi_1^{m\ell}})^*(\psi_p))^{N-3-k} \cdot 1/m\ell$$

is a nonincreasing function of k , so

$$\Theta_k = \lim_{m \rightarrow \infty} ((\overline{\pi_2^{m\ell}})^*(\psi_p))^k \cdot ((\overline{\pi_1^{m\ell}})^*(\psi_p))^{N-3-k} \cdot 1/m\ell$$

is a nonincreasing function of k . \square

4. Appendix: Birational invariance of dynamical degrees

Dynamical degrees of rational *maps* are birational invariants ([DS05, DS08, CCLG10, Tru15]). It has been known to experts that the same is true for rational correspondences. However, this statement cannot be found in the literature (see Remark 2.9). We present a detailed account of how the proofs in [Tru15] can be modified for a complete proof of Theorem 2.8. We closely follow [Tru15] and keep similar notation.

4.1 Some definitions

DEFINITION 4.1. A cycle class $\alpha \in A_k(X)$ (or $A^k(X)$ for X smooth) is called **effective** if it can be written as $\sum_V \alpha_V [V]$ with all $\alpha_V \geq 0$.

DEFINITION 4.2. For $\alpha_1, \alpha_2 \in A_k(X)$ (or $A^k(X)$ for X smooth), we write $\alpha_1 \geq \alpha_2$ if $\alpha_1 - \alpha_2$ is effective.

DEFINITION 4.3. For X a variety, $U \subseteq X$ an open set, and a cycle $\alpha = \sum_V \alpha_V [V] \in \mathcal{Z}_k(U)$, we define the **closure** $\overline{\alpha}$ of α in X to be $\sum_V \alpha_V [\overline{V}] \in \mathcal{Z}_k(X)$.

DEFINITION 4.4. For X a variety, \mathfrak{h} an ample divisor on X , and $V \subseteq X$ a k -dimensional subvariety, the degree $\deg(V)$ of V is the intersection number $\mathfrak{h}^k \cdot V$.

Note that if $[V_1] = [V_2]$ in $A_k(X)$, then $\deg(V_1) = \deg(V_2)$. Thus for $\alpha = \sum_V \alpha_V [V]$ in $A_k(X)$, we may define $\deg(\alpha) = \sum_V \alpha_V \deg(V)$.

For the rest of this section, fix $(\Gamma, \pi_X, \pi_Y) : X \rightrightarrows Y$ a dominant rational correspondence. Without loss of generality we may assume Γ is smooth and projective. Over a dense open set of Y , the fibers of π_Y have dimension $\dim X - \dim Y$. Over a dense open set of X , π_X is a covering map. Thus there is a dense open set Γ_\circ of Γ with $\pi_{X,\circ} := \pi_X|_{\Gamma_\circ}$ a covering map of its image

(thus proper), and such that $\pi_{Y,\circ} := \pi_Y|_{\Gamma_\circ}$ has fibers of the correct dimension $\dim X - \dim Y$. Since Y is smooth and Γ is smooth (in particular Cohen-Macaulay), $\pi_{Y,\circ}$ is flat.

For $W \subseteq Y$ a subvariety that intersects every irreducible component of $Y \setminus (\pi_{Y,\circ}(\Gamma_\circ))$ properly, $\pi_Y^*([W]) = [\pi_Y^{-1}(W)]$, and so the pullback $\pi_Y^*([W])$ is effective, and well-defined as a cycle on Γ , not merely as a cycle class. For such W , the pullback $[\Gamma]^*([W]) = (\pi_X)_* \pi_Y^*([W])$ is likewise effective, and well-defined as a cycle on X . (See Lemma 3.1 of [Tru15].)

DEFINITION 4.5. For a subvariety W of Y , the pullback $\pi_{Y,\circ}^*[W]$ is a cycle in Γ_\circ . By taking its closure we obtain a cycle in Γ that we denote $\pi_Y^\circ[W]$. Thus we have maps

$$\pi_Y^\circ : \mathcal{Z}_k(Y) \rightarrow \mathcal{Z}_{\dim X - \dim Y + k}(\Gamma)$$

and

$$\Gamma^\circ := (\pi_X)_* \circ \pi_Y^\circ : \mathcal{Z}_k(Y) \rightarrow \mathcal{Z}_{\dim X - \dim Y + k}(X).$$

Equivalently, for $\beta \in \mathcal{Z}_k(Y)$ set

$$\Gamma^\circ(\beta) = \overline{(\pi_{X,\circ})_* \circ \pi_{Y,\circ}^*(\beta)}.$$

Remark 4.6. The map Γ° is modeled on the *proper transform* $g^\circ : \mathcal{Z}_k(Y) \rightarrow \mathcal{Z}_{\dim X - \dim Y + k}(X)$ induced by $g : X \dashrightarrow Y$ in the notation and terminology of [Tru15].

Remark 4.7. Note that for a subvariety $W \subseteq Y$, $\Gamma^\circ([W])$ is effective.

LEMMA 4.8 (Lemma 3.2a of [Tru15]). *Fix an embedding of Y into projective space \mathbb{P}^M , and denote by \mathfrak{h}_Y the pullback to Y of the hyperplane class. Let $k = 0, \dots, \dim Y$, and let $V \subseteq X$ be a proper subvariety. Then there is a linear subspace $H^k \subseteq \mathbb{P}^M$ of codimension k such that H^k intersects Y properly, and $[\Gamma]^*([H^k|_Y])$ is effective, well-defined as a cycle on X , and has no irreducible component supported on V . In particular, $[\Gamma]^*(\mathfrak{h}_Y^k) \in A^k(X)$ is effective.*

Proof. By Lemma 3.2 in [Tru15], for each of the finitely many components Γ_i of Γ , the statement holds for the restriction $\pi_Y|_{\Gamma_i} : \Gamma_i \rightarrow Y$. By additivity, the statement holds for $\pi_Y : \Gamma \rightarrow Y$. Since pushforwards preserve effectiveness, the statement also holds for $[\Gamma]^* = (\pi_X)_* \circ \pi_Y^*$. \square

4.2 Proof of Theorem 2.8

LEMMA 4.9 (Lemma 3.4 of [Tru15]). *Let Y be a smooth projective variety with an embedding $Y \hookrightarrow \mathbb{P}^M$. Denote by \mathfrak{h}_Y the pullback of the hyperplane class. Then there is a constant $C > 0$ such that for all rational correspondences $(\Gamma, \pi_X, \pi_Y) : X \dashrightarrow Y$ and for all codimension k subvarieties $W \subseteq Y$,*

$$\Gamma^\circ([W]) \leq C \deg(W)([\Gamma]^*(\mathfrak{h}_Y^k))$$

in $A^k(X)$, where Γ° is as in Definition 4.5.

Proof. By Lemma 3.4 in [Tru15], for each of the finitely many components Γ_i of Γ , the statement holds for the restriction $\pi_Y|_{\Gamma_i} : \Gamma_i \rightarrow Y$. By additivity, the statement holds for $\pi_Y : \Gamma \rightarrow Y$. Since pushforwards preserve effectiveness, the statement also holds for $[\Gamma]^* = (\pi_X)_* \circ \pi_Y^*$. \square

LEMMA 4.10 (Lemma 3.5 of [Tru15]). *Let Y and Z be smooth projective varieties. We fix embeddings $Y \hookrightarrow \mathbb{P}^M$ and $Z \hookrightarrow \mathbb{P}^N$, and denote by \mathfrak{h}_Y and \mathfrak{h}_Z the hyperplane classes on Y and Z respectively. Then there is a constant $C > 0$ such that for all dominant rational correspondences $(\Gamma, \pi_X, \pi_Y) : X \dashrightarrow Y$ and $(\Gamma', \pi'_X, \pi'_Z) : Y \dashrightarrow Z$, we have in $A^k(X)$:*

$$[\Gamma' \circ \Gamma]^*(\mathfrak{h}_Z^k) \leq C \deg([\Gamma']^* \mathfrak{h}_Z^k) \cdot ([\Gamma]^* \mathfrak{h}_Y^k).$$

Proof. We follow the proof of Lemma 3.5 in [Tru15], making modifications as necessary. As in Section 4.1, we can find dense open subsets Γ_\circ of Γ and Γ'_\circ of Γ' such that

- $\pi_{X,\circ} = \pi_X|_{\Gamma_\circ}$ and $\pi'_{Y,\circ} = \pi'_Y|_{\Gamma'_\circ}$ are covering maps (hence proper maps) of dense open sets in X and Y respectively,
- $\pi_{Y,\circ} = \pi_Y|_{\Gamma_\circ}$ and $\pi'_{Z,\circ} = \pi'_Z|_{\Gamma'_\circ}$ have fibers of the correct dimensions $\dim X - \dim Y$ and $\dim Y - \dim Z$, respectively, thus are flat maps, and
- $\pi_{Y,\circ}(\Gamma_\circ) \subseteq \pi'_{Y,\circ}(\Gamma'_\circ)$.

As in Definition 2.5, we set

$$\Gamma' \circ \Gamma = (\Gamma_\circ)_{\pi_{Y,\circ}} \times_{\pi'_{Y,\circ}} (\Gamma'_\circ).$$

Denote by pr and pr' the maps from $\Gamma' \circ \Gamma$ to Γ and Γ' , respectively. Choose a smooth projective compactification $\overline{\Gamma' \circ \Gamma}$ of $\Gamma' \circ \Gamma$, extending the maps pr and pr' to $\overline{\text{pr}}$ and $\overline{\text{pr}'}$ respectively.

We have a diagram

$$\begin{array}{ccccc}
 & & \Gamma' \circ \Gamma & & \\
 & \swarrow \text{pr} & & \searrow \text{pr}' & \\
 & \Gamma_\circ & & \Gamma'_\circ & \\
 \swarrow \pi_{X,\circ} & & \searrow \pi_{Y,\circ} & \swarrow \pi'_{Y,\circ} & \searrow \pi'_{Z,\circ} \\
 X & & Y & & Z
 \end{array}$$

and a diagram of compactifications

$$\begin{array}{ccccc}
 & & \overline{\Gamma' \circ \Gamma} & & \\
 & \swarrow \overline{\text{pr}} & & \searrow \overline{\text{pr}'} & \\
 & \overline{\Gamma} & & \overline{\Gamma'} & \\
 \swarrow \pi_X & & \searrow \pi_Y & \swarrow \pi'_Y & \searrow \pi'_Z \\
 X & & Y & & Z
 \end{array}$$

For $\gamma \in \mathcal{Z}_k(Z)$, set

$$(\Gamma')^\circ(\gamma) = \overline{(\pi'_{Y,\circ})_* \circ (\pi'_{Z,\circ})^*(\gamma)}$$

and

$$(\Gamma' \circ \Gamma)^\circ(\gamma) = \overline{(\pi_{X,\circ} \circ \text{pr})_* \circ (\pi'_{Z,\circ} \circ \text{pr}')^*(\gamma)}$$

For $\beta \in \mathcal{Z}_k(Y)$, set

$$\Gamma^\circ(\beta) = \overline{(\pi_{X,\circ})_* \circ (\pi_{Y,\circ})^*(\beta)}$$

By Lemma 4.8, there is a linear subspace $H^k \subseteq \mathbb{P}^N$ intersecting Z properly, such that

- (i) $[\Gamma' \circ \Gamma]^*([H^k|_Z])$ is effective, well-defined as a cycle, and has no irreducible component supported on $X \setminus (\pi_{X,\circ}(\text{pr}(\Gamma' \circ \Gamma)))$, and
- (ii) $[\Gamma']^*([H^k|_Z])$ is effective and well-defined as a cycle.

Then by (i),

$$[\Gamma' \circ \Gamma]^*(\mathfrak{h}_Z^k) = [\Gamma' \circ \Gamma]^*([H^k|_Z]) = (\Gamma' \circ \Gamma)^\circ([H^k|_Z]).$$

This by definition is the closure of

$$\begin{aligned} (\pi_{X,\circ} \circ \text{pr})_*(\pi'_{Z,\circ} \circ \text{pr}')^*([H^k|_Z]) &= (\pi_{X,\circ})_* \circ \text{pr}_* \circ (\text{pr}')^* \circ (\pi'_{Z,\circ})^*([H^k|_Z]) \\ &= (\pi_{X,\circ})_* \circ (\pi_{Y,\circ})^* \circ (\pi'_{Y,\circ})_* \circ (\pi'_{Z,\circ})^*([H^k|_Z]). \end{aligned}$$

The second equality follows from the commutativity of flat pullback and proper pushforward in a cartesian square ([Ful98], Proposition 1.7). Thus

$$\begin{aligned} [\Gamma' \circ \Gamma]^*(\mathfrak{h}_Z^k) &= \Gamma^\circ((\Gamma')^\circ([H^k|_Z])) \\ &\leq \Gamma^\circ([\Gamma']^*([H^k|_Z])) \quad (\text{by (ii)}) \\ &\leq C \deg([\Gamma']^*(\mathfrak{h}_Z^k)) \cdot ([\Gamma]^*(\mathfrak{h}_Y^k)), \end{aligned}$$

by Lemma 4.9. □

THEOREM 2.8. *Let $(\Gamma, \pi_1, \pi_2) : X \dashrightarrow X$ be a dominant rational self-correspondence, and let $\chi : X \dashrightarrow X'$ be a birational equivalence. We obtain a dominant rational self-correspondence on X' through conjugation by χ as follows. Let U be the domain of definition of χ , and set $\Gamma' = \pi_1^{-1}(U) \cap \pi_2^{-1}(U)$. We have a dominant rational self-correspondence*

$$(\Gamma', \chi \circ \pi_1, \chi \circ \pi_2) : X' \dashrightarrow X'.$$

Then the dynamical degrees of Γ and Γ' are equal.

Proof. Fix ample divisors \mathfrak{h} on X and \mathfrak{h}' on X' . For every n , we have

$$\begin{array}{ccc} X & \xrightarrow{\Gamma^n} & X \\ \chi \downarrow & & \downarrow \chi \\ X' & \xrightarrow{(\Gamma')^n} & X' \end{array}$$

Applying Lemma 4.10 twice,

$$\begin{aligned} [\Gamma^n]^*(\mathfrak{h}^k) &= [\chi^{-1} \circ (\Gamma')^n \circ \chi]^*(\mathfrak{h}^k) \\ &\leq C_1 \deg([\chi^{-1}]^*(\mathfrak{h}^k)) \cdot [(\Gamma')^n \circ \chi]^*((\mathfrak{h}')^k) \\ &\leq C_1 C_2 (\deg([\chi^{-1}]^*(\mathfrak{h}^k))) (\deg([(\Gamma')^n]^*((\mathfrak{h}')^k))) ([\chi]^*((\mathfrak{h}')^k)), \end{aligned}$$

where $C_1, C_2 > 0$. Thus

$$\begin{aligned} \deg([\Gamma^n]^*(\mathfrak{h}^k)) &\leq \left(C_1 C_2 (\deg([\chi^{-1}]^*(\mathfrak{h}^k))) (\deg([\chi]^*((\mathfrak{h}')^k))) \right) (\deg([(\Gamma')^n]^*((\mathfrak{h}')^k))) \\ &= C \deg([(\Gamma')^n]^*((\mathfrak{h}')^k)). \end{aligned}$$

Taking n th roots and then the limit as $n \rightarrow \infty$, the constants disappear, and

$$(k\text{th dynamical degree of } \Gamma) \leq (k\text{th dynamical degree of } \Gamma').$$

By symmetry,

$$(k\text{th dynamical degree of } \Gamma) = (k\text{th dynamical degree of } \Gamma').$$

□

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